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Generalisation of Wigner's theorem for dissipative quantum systems

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Abstract. We present a quantum Langevin equation in the Schrödinger picture. The friction term is motivated by a natural generalisation of Wigner's theorem or, equivalently, of Dirac's superposition principle. We apply the model to spin relaxation and to the Brownian motion of the harmonic oscillator. In both cases the state vectors evolve asymptotically to a distribution which has the Gibbs state as corresponding density matrix. We show that for any initial state, the harmonic oscillator tends to a coherent state.

1. Introduction

The classical Langevin equation [1] describes a point particle undergoing frictional and stochastic forces in addition to deterministic conservative forces. The physical origin of the friction and fluctuations appear in the equation only via some constants depending on the macroscopic state of the environment. One generally believes that the non-Hamiltonian terms can be derived from a reduced description of the evolution of a larger isolated system. In some particular cases such dilation schemes have been made rigorous (see, e.g., [2]).

Our aim in the present paper is to propose a quantum Langevin equation in the Schrödinger picture. That is to say, an evolution equation for the state vector which contains deterministic non-unitary frictional terms and stochastic unitary fluctuation terms in addition to the Hamiltonian term. In order to describe the friction we use a natural generalisation of Wigner's theorem. The friction term so obtained is not unitary, but preserves the norm $\|\psi_t\|$ and the superpositions (with possibly time-dependent coefficients). The fluctuations are described by the same tools as in classical mechanics, namely by stochastic differential equations (SDE). The noise may thus be termed classical noise.

The two canonical examples are studied, namely the harmonic oscillator and the spin-$\frac{1}{2}$. In both cases the state vectors evolve asymptotically to a distribution which has the Gibbs state as corresponding density matrix. The temperatures of these Gibbs states, however, differ from the ones obtained for a classical oscillator, or a classical magnetic moment.

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2. Quantum Langevin equation

The unitary evolutions on a Hilbert space $\mathscr{H}$, $\psi_t = U_t\psi_0$, $\dot{\psi}_t = -\mathrm{i}H\psi_t$ $(H^\dagger = H)$, are characterised by the following properties:

(a) U_t is bijective,

(b1) $|\langle\psi|\phi\rangle| = |\langle U_t\psi|U_t\phi\rangle|$,

(c) $U_t \circ U_s = U_{t+s}$, $s\text{-}\lim_{t\to 0} U_t = 1$ and $\|U_t\psi\| = \|\psi\|$.

This is the content of Wigner's theorem [3]. It is known that if dim $\mathscr{H} \geq 3$, then (b1) is equivalent to [4]

(b2) $\psi \perp \phi \Leftrightarrow U_t\psi \perp U_t\phi$.

We shall need the following result.

Lemma. Any map $U_t: \mathscr{H} \to \mathscr{H}$ (t fixed) which satisfies (a) and (b2) preserves the superpositions in the following sense:

(d) If $\psi = \Sigma\, \alpha_i\phi_i$ (with $\alpha_i \in \mathbb{C}$, $\phi_i \in \mathscr{H}$), then there are complex numbers β_i such that $U_t\psi = \Sigma\, \beta_i U_t\phi_i$

Proof. From (a) and (b2) it follows immediately that U_t preserves subsets of $\mathscr{H}$ that are their own second orthocomplement, i.e.

(e) $M^{\perp\perp} = M \Leftrightarrow (U_tM)^{\perp\perp} = U_tM$

where $M^\perp = \{\psi \in \mathscr{H} | \psi \perp \phi \forall \phi \in M\}$, $U_tM = \{U_t\psi | \psi \in M\}$. Now a subset M satisfying $M = M^{\perp\perp}$ is nothing but a closed linear subspace of $\mathscr{H}$ [5]. Hence, any vector ψ in the subspace generated by the ϕ_i's is mapped on a vector in the subspace generated by the $U_t\phi_i$'s.

In [6] we generalised Wigner's theorem by proving the following.

Theorem. Any evolution on a Hilbert space with dim $\mathscr{H} \geq 3$ which satisfies (a), (c) and (d), or equivalently (a), (c) and (e), but not necessarily (b), is of the form $\psi_t = V_t\psi_0 / \|V_t\psi_0\|$ where V_t is a semigroup of linear contracting operators (i.e. $\|V_t\psi\| \leq \|\psi\|$)†.

If $Z = -\mathrm{i}H - B$ $(H^\dagger = H, B^\dagger = B)$ is the generator of V_t, one obtains the following evolution equation:

$$\dot{\psi}_t = -\mathrm{i}H\psi_t + (\langle B\rangle_{\psi_t} - B)\psi_t. \tag{1}$$

Let us recall that any contraction semigroup can be dilated to a unitary evolution U_t on a larger Hilbert space $\mathscr{K}$: $V_t = PU_tP$ with $P\mathscr{K} = \mathscr{H}$ [7, 8]. The evolution (1) can thus be regarded as a reduced description [9].

Here we are mainly interested in the case $B = kH$ $(k > 0)$, for which one obtains

$$(\mathrm{d}/\mathrm{d}t)\langle H\rangle_{\psi_t} = -2k(\langle H^2\rangle_{\psi_t} - \langle H\rangle^2_{\psi_t}) \leq 0.$$

The system thus dissipates energy, except when it is in an eigenstate of H, and tends asymptotically to an eigenstate of H [10]. The latter are thus analogous to limit circles. Note that only the ground state is stable‡.

† Assumption (a) can be weakened to: U_t is one-to-one and $U_t\mathscr{H}$ is a closed subspace of $\mathscr{H}$.

‡ From a semiclassical point of view one imagines that an electron in a state at rest (i.e. an eigenstate of H) does not radiate. On the other hand, the non-stationary solutions correspond to states which move, and thus radiate. From this point of view one may think of the states at rest as the results of evolutionary processes, described for instance by equation (1).

In the following we consider the SDE obtained by adding classical fluctuations to (1) ($\mathscr{H}=\mathscr{L}^2(\mathbb{R}^n)\otimes\mathbb{C}^m$)†:

$$d\psi_t = -iH\psi_t dt + k(\langle H\rangle_{\psi_t} - H)\psi_t dt - i\boldsymbol{q}(\psi_t \circ d\boldsymbol{\omega}_t) - i\boldsymbol{p}(\psi_t \circ d\boldsymbol{\varepsilon}_t) - i\boldsymbol{S}(\psi_t \circ d\boldsymbol{b}_t) \qquad (2)$$

where $H = H^\dagger$ is the Hamiltonian; $\boldsymbol{q}$, $\boldsymbol{p}$, $\boldsymbol{S}$ are the usual position, momentum and spin operators; $(\boldsymbol{\omega}_t, \boldsymbol{\varepsilon}_t, \boldsymbol{b}_t)$ is the $(2n+3)$-dimensional Wiener process; $\boldsymbol{q}\cdot(\psi_t \circ d\boldsymbol{\omega}_t) \equiv \Sigma_{l=1}^n q_l\psi_t \circ d\omega_{lt}$ and similarly for the $\boldsymbol{p}$ and $\boldsymbol{S}$ terms; and $\circ$ denotes the Stratonovich product [1, 12]‡ which is related to Ito's product by $x_t \circ dy_t = x_t dy_t + \frac{1}{2}dx_t dy_t$. It is straightforward to verify that (2) preserves the norm of ψ_t (i.e. $d\langle\psi_t|\psi_t\rangle = 0$). This is not true if we use Itô's product in (2), hence our choice of the Stratonovich product.

An interesting property of (2) is the following: if $\psi_0 = \alpha_0\chi_0 + \beta_0\phi_0$ ($\alpha_0, \beta_0 \in \mathbb{C}$; $\chi_0, \phi_0 \in \mathscr{H}$), then§

$$\psi_t = \alpha_t\chi_t + \beta_t\phi_t$$

where ψ_t, χ_t, ϕ_t are the solutions of (2) corresponding to the initial condition ψ_0, χ_0, ϕ_0; and the complex coefficients satisfy

$$d\alpha_t = k\alpha_t(\langle H\rangle_{\psi_t} - \langle H\rangle_{\chi_t})\, dt \qquad d\beta_t = k\beta_t(\langle H\rangle_{\psi_t} - \langle H\rangle_{\phi_t})\, dt.$$

Accordingly, (2) preserves the superpositions, but with possibly time dependent coefficients. In fact, any evolution satisfying this generalised superposition principle is of the form (1) with possibly time dependent operators H and B [6].

3. Examples

Our first example is the Brownian motion of a damped spin-$\frac{1}{2}$. Let $\mathscr{H} = \mathbb{C}^2$ and $H = \frac{1}{2}\omega\sigma_z$, with obvious notations. Consider the following SDE:

$$d\psi_t = -iH\psi_t\, dt + k(\langle H\rangle_{\psi_t} - H)\psi_t\, dt - \tfrac{1}{2}i\boldsymbol{\sigma}\psi_t \circ d\boldsymbol{b}_t \qquad (3)$$

where the three-dimensional Wiener process $\boldsymbol{b}_t$ satisfies $db_{kt}\, db_{lt} = \delta_{kl}D_k\, dt$. For symmetry reasons we put $D_x = D_y \equiv D_\perp$. Let $\boldsymbol{m}_t = \langle\boldsymbol{\sigma}\rangle_{\psi_t}$. From (3) one obtains‖:

$$d\boldsymbol{m}_t = \omega(\boldsymbol{e}_z \wedge \boldsymbol{m}_t + k(\boldsymbol{e}_z \wedge \boldsymbol{m}_t) \wedge \boldsymbol{m}_t)dt - \boldsymbol{m}_t \mathbin{\mathring{\wedge}} d\boldsymbol{b}_t \qquad (4)$$

with $\boldsymbol{e}_z = (0, 0, 1)$. Equation (4) has been studied in [13]. If the magnetic energy is small compared with the thermal energy, which is the usual condition, one may linearise (4). This way one recovers the well known phenomenological Bloch equations [14], with $T_1 = (2D_\perp)^{-1}$ and $T_2 = (D_\perp + D_z)^{-1}$ the longitudinal and transversal relaxation times.

The relation between $\boldsymbol{m}_t$ and ψ_t can be inverted:

$$\psi_t\psi_t^\dagger = \tfrac{1}{2}(1 + \boldsymbol{m}_t\boldsymbol{\sigma}).$$

† Since $\mathscr{H}$ is infinite dimensional the mathematical meaning of (2) involves stochastic analysis in infinite dimensions. See for instance [11]. In the examples we shall consider in the next section, however, either $\mathscr{H}$ is finite dimensional (spin-$\frac{1}{2}$) or, thanks to the completeness of the coherent states of the harmonic oscillator, (2) can be defined using only finite dimensional stochastic analysis.

‡ The Stratonovich product is defined such that $d(x_t y_t) = x_t \circ dy_t + y_t \circ dx_t$. Hence, intuitively $\psi_t \circ d\omega_t \approx \psi_t\dot{\omega}(t)\, dt$ with ω a very wiggly function.

§ The following is an equality between random variables. It holds thus for all realisations, except a set of measure zero.

‖ Where $\mathring{\wedge}$ denotes the Stratonovich wedge product. For instance $(\boldsymbol{m} \mathbin{\mathring{\wedge}} d\boldsymbol{b})_x = m_y \circ db_z - m_z \circ db_y = (\boldsymbol{m} \wedge d\boldsymbol{b})_x - \frac{1}{2}(D_y + D_z)m_x$.

Consequently (3) and (4) are equivalent. We do not know the time dependent solution of (4). However its asymptotic equilibrium distribution is easily found

$$\mu_\infty(\boldsymbol{m}) = N\exp(-\gamma\omega m_z/2)\delta(|\boldsymbol{m}|-1)$$

with $\gamma = 4k/D_\perp$ and N a normalisation constant. The corresponding distribution $\lambda_\infty(\psi)$ of the state vector and the density matrix ρ_∞ are†:

$$\lambda_\infty(\psi) = N\exp(-\gamma\langle H\rangle\psi)\delta(\|\psi\|-1)$$

$$\rho_\infty = \int \tfrac{1}{2}(1+\boldsymbol{m}\cdot\boldsymbol{\sigma})\mu_\infty(\boldsymbol{m})\mathrm{d}^3m = \mathrm{e}^{-\beta H}/\mathrm{Tr}(\mathrm{e}^{-\beta H})$$

where the natural temperature β satisfies

$$\tanh(\beta\cdot\tfrac{1}{2}\omega) = \coth(2k\omega/D_\perp) - D_\perp/2k\omega$$

i.e. $\beta = \frac{1}{3}\gamma + \mathrm{O}(k/D_\perp)^3$.

Let us now consider the Brownian motion of a damped quantum harmonic oscillator. Let $\mathscr{H} = \mathscr{L}^2(\mathbb{R})$, $H = \Omega a^\dagger a$ (with $[a, a^\dagger] = 1$) and

$$\mathrm{d}\psi_t = -\mathrm{i}H\psi_t\mathrm{d}t + k(\langle H\rangle_{\psi_t} - H)\psi_t\mathrm{d}t - \mathrm{i}q\psi_t\circ\mathrm{d}\omega_t - \mathrm{i}p\psi_t\circ\mathrm{d}\varepsilon_t \tag{5}$$

with $(\mathrm{d}\omega_t)^2 = D_q\mathrm{d}t$, $(\mathrm{d}\varepsilon_t)^2 = D_p\mathrm{d}t$, $\mathrm{d}\omega_t\mathrm{d}\varepsilon_t = 0$. In order to simplify the study of (5) we shall use the completeness of the set of coherent states and the fact that (5) preserves the superpositions. For any complex number α let us denote $|\alpha\rangle$ the coherent state defined by $a|\alpha\rangle = \alpha|\alpha\rangle$. Let α_t be a solution of the following SDE:

$$\mathrm{d}\alpha_t = -\Omega(\mathrm{i}+k)\alpha_t\mathrm{d}t + 2^{-1/2}(\mathrm{d}\varepsilon_t - \mathrm{i}\mathrm{d}\omega_t). \tag{6}$$

A straightforward computation shows that whenever $a|\alpha_t\rangle = \alpha_t|\alpha_t\rangle$, then $\mathrm{d}(a|\alpha_t\rangle - \alpha_t|\alpha_t\rangle) = 0$, where $\mathrm{d}|\alpha_t\rangle$ and $\mathrm{d}\alpha_t$ are computed with (5) and (6) respectively. Consequently if ψ_0 is a coherent state, then the solution ψ_t of (5) is a coherent state-valued stochastic process, which can be computed with the help of the well known equation (6). Now, any state ψ_0 is a superposition of coherent states. Thus we have for the general solution of (5):

$$\psi_t = (\|\psi_0\|/\|\phi_t\|)\cdot\phi_t$$

$$\phi_t = \int_{\mathbb{C}} \mathrm{d}z\langle z|\psi_0\rangle\exp\left(-k\int_0^t |\alpha_s(z)|^2\,\mathrm{d}s\right)|\alpha_t(z)\rangle$$

where $\alpha_t(z)$ is the solution of (6) with initial condition $\alpha_0(z) = z$.

Let θ_t denote the solution of (6) with initial condition $\theta_0 = 0$, i.e. $\theta_t = \alpha_t(0)$. One has

$$\mathrm{d}(|\alpha_t(z)|^2 - |\theta_t|^2) = -2k\Omega(|\alpha_t(z)|^2 - |\theta_t|^2)\mathrm{d}t \leqslant 0$$

for all $z\in\mathbb{C}$. All the stochastic processes $|\alpha_t(z)\rangle$ tend thus to $|\theta_t\rangle$. Consequently‡

$$\psi_t \xrightarrow{t\to\infty} |\theta_t\rangle\ \forall\psi_0\in\mathscr{H}.$$

The asymptotic distribution $\Lambda_\infty(\psi)$ of (5) is thus concentrated on the coherent states and can be obtained from the well known solution of (6) [1]:

$$\Lambda_\infty(\psi) = N\exp[-(2k/D)\langle H\rangle_\psi]\delta(\|\psi\|-1)\delta(\langle a^\dagger a\rangle_\psi - \langle a\rangle_\psi\cdot\langle a^\dagger\rangle_\psi) \tag{7}$$

† Note that since the evolution (1) is nonlinear, the density matrix does not satisfy any closed evolution equation.

‡ We conjecture that this result holds for all Hamiltonians, where the initial state of $|\theta_t\rangle$ is the ground state.

where, for simplicity, we assume $D_p = D_q = D$. The corresponding density matrix is

$$\rho_\infty = \int_{\mathbb{C}} |\alpha\rangle\langle\alpha|\Lambda_\infty(|\alpha\rangle)\,d\alpha$$
$$= e^{-\beta H}/\mathrm{Tr}(e^{-\beta H}) \tag{8}$$

with β satisfying $e^\beta = 1 + 2k/D$.

4. Conclusion

With a natural generalisation of Wigner's theorem we motivated the study of a dissipative Schrödinger equation. The friction term applies as well to spins as to the spatial degrees of freedom (or to any quantum system described by a Hilbert space). This non-unitary evolution can be dilated to a unitary evolution on a larger Hilbert space. We thus make contact with the framework of reduced descriptions, which is much used in statistical mechanics [8, 15]. The difference with the traditional approaches, like the Pauli master equation, comes from the fact that in our case the reduction is done by a pure state preserving projection, in opposition to the partial trace projection [9]. Let us emphasise that this non-unitary evolution preserves the superpositions, but with time-dependent complex coefficients. In fact our proposal is the most general evolution which satisfies this generalisation of Dirac's superposition principle [6].

By adding fluctuations in the standard way, we obtain models of the Brownian motion of the damped quantum harmonic oscillator and the damped spin-$\frac{1}{2}$, with irreversible approaches to the equilibrium Gibbs states. In both models classical aspects arise, though no violation of the quantum principles, like the uncertainty principle, can occur since we are still in the framework of Hilbert space quantum kinetics. The quantum oscillator tends for large times to a coherent state and its mean position and momentum follow then the law of the Brownian motion of a damped classical oscillator†. Moreover, in the high-temperature limit, the natural temperatures of the quantum and classical Gibbs state (see (7) and (8)) become equal.

At this point one should mention the highly interesting work by R L Hudson, K R Parthasarathy, R F Streater and others on non-commutative noise [16-18]. The objectives of this theory are somewhat similar to ours, but with some important differences. In particular, in this approach the system does not remain in a probabilistic mixture of pure states, but gets correlated to its environment. The atomic polarisation correlation experiments of Clauser, Aspect and others [19, 20] give overwhelming evidence that such systems exist, even when spatially separated. However in these experiments the systems must be very well isolated. Indeed any perturbation has the effect of suppressing (or reducing if the perturbation is very weak) the violation of Bell's inequality [19]. Moreover, after the systems have interacted with the measurement apparatuses, a dissipative interaction of course, the systems are separated. Hence these experiments are not in contradiction with the assumption that there exist dissipative interactions that cause a continuous destruction of the phase relation between the system and its environment. On the other hand such interactions remain hypothetical.

In recent years a lot of effort has been directed to the study of friction in quantum mechanics (see, e.g., [21-23]). Most approaches are restricted to wave mechanics, i.e.

† This suggests the idea of a dynamical classical limit.

$\mathscr{H} = L^2(\mathbb{R}^n)$, and deal with nonlinear real potentials which are required to quantise the classical friction term proportional to minus the velocity. In fact, to the best of our knowledge, our proposal is the only one compatible with the Hilbert space structure of quantum kinematics.

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